

On the Hyperhomology of the Small Gobelin in Codimension 2

X. Gómez-Mont* and L. Núñez-Betancourt†

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Abstract

Given a zero-dimensional Gorenstein algebra \mathbb{B} and two syzygies between two elements $f_1, f_2 \in \mathbb{B}$, one constructs a double complex of \mathbb{B} -modules, $\mathcal{G}_{\mathbb{B}}$, called the small Gobelin. We describe an inductive procedure to construct the even and odd hyperhomologies of this complex. For high degrees, the difference $\dim \mathbb{H}_{j+2}(\mathcal{G}_{\mathbb{B}}) - \dim \mathbb{H}_j(\mathcal{G}_{\mathbb{B}})$ is constant, but possibly with a different value for even and odd degrees. We describe two flags of ideals in \mathbb{B} which codify the above differences of dimension. The motivation to study this double complex comes from understanding the tangency condition between a vector field and a complete intersection, and invariants constructed in the zero locus of the vector field $\text{Spec}(\mathbb{B})$.

Introduction

Let K be a field of characteristic 0 and \mathcal{O} a local Noetherian K -algebra. Consider a matrix identity over \mathcal{O}

$$\begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{\ell 1} & \cdots & \varphi_{\ell N} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{\ell 1} & \cdots & c_{\ell k} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} \quad (1)$$

and write this equation as

$$\varphi X = cf. \quad (2)$$

In particular if φ is the derivative Df of a differentiable function f , the identity expresses the fact that the vector field X is tangent to the variety defined by $f = 0$.

Bothmer, Ebeling and the first author have constructed in [1] from the matrix identity (2) a double complex of free \mathcal{O} -modules, called the Gobelin, obtained by weaving Buchsbaum-Eisenbud and Koszul complexes as strands, and that under suitable hypothesis can be used to compute the homology of the complex of $\frac{\mathcal{O}}{\text{Im}(f^*)}$ -modules:

$$0 \longleftarrow \frac{\mathcal{O}}{\text{Im}(f^*)} \xleftarrow{X^*} \frac{\mathcal{O}^{\oplus N}}{\text{Im}(\varphi^*)} \otimes_K \frac{\mathcal{O}}{\text{Im}(f^*)} \xleftarrow{X^*} \frac{\Lambda^2 \mathcal{O}^{\oplus N}}{\Lambda^2 \text{Im}(\varphi^*)} \otimes_K \frac{\mathcal{O}}{\text{Im}(f^*)} \xleftarrow{X^*} \cdots \quad (3)$$

Similarly, they also constructed in [1] another double complex $\mathcal{G}_{\mathbb{B}}^{\ell}$, called the small Gobelin (see [1] or Figure 1), which is quasi-isomorphic to the Gobelin, but formed with free $\mathbb{B} := \frac{\mathcal{O}}{\text{Im}(X^*)}$ -modules. The homology of (3) has certain periodicity properties when $k = \ell = 1$ (see [5]), and the objective of this work is to use the small Gobelin to set up an inductive procedure on ℓ to see how these periodicity properties generalize. In this paper, we carry this out for $k = 2$ and $\ell = 1, 2$.

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If $\mathcal{G}_{\mathbb{B}}^{\ell-1}$ denotes the small Gobel in formed with the equation (1) by deleting the last row on the matrices in the equation, we obtain in Theorem 2.1 an exact sequence of double complexes

$$0 \longrightarrow \mathcal{G}_{\mathbb{B}}^{\ell-1} \xrightarrow{i} \mathcal{G}_{\mathbb{B}}^{\ell} \xrightarrow{\sigma^*} \mathcal{G}_{\mathbb{B}}^{\ell}(-1, -1) \longrightarrow 0$$

giving rise to a long exact sequence of hyperhomology groups

$$\dots \longrightarrow \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^{\ell-1}) \xrightarrow{\iota} \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^{\ell}) \xrightarrow{\sigma^*} \mathbb{H}_{j-2}(\mathcal{G}_{\mathbb{B}}^{\ell}) \xrightarrow{\partial} \mathbb{H}_{j-1}(\mathcal{G}_{\mathbb{B}}^{\ell-1}) \longrightarrow \dots$$

This long exact sequence relates the hyperhomology groups $\mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^{\ell})$ with $\mathbb{H}_{j-2}(\mathcal{G}_{\mathbb{B}}^{\ell})$ with kernels and cokernels in the hyperhomology groups of $\mathcal{G}_{\mathbb{B}}^{\ell-1}$, and this is the basis of our inductive procedure.

From Section 3 onwards, we restrict to the case where \mathcal{O} is an N dimensional Gorenstein K -algebra (in particular a local regular ring or the ring of power series over K (formal or convergent) or a complete intersection), assume given a matrix identity (1) with $k = \ell = 2$, that X_1, \dots, X_N is an \mathcal{O} -regular sequence and denote by $\mathbb{B} := \frac{\mathcal{O}}{\text{Im}(X^*)}$. It is a 0 dimensional Gorenstein ring, which is a finite dimensional K -vector space. The relation (1) becomes in \mathbb{B} :

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (4)$$

The total complex $\text{tot}(\mathcal{G}_{\mathbb{B}}^2)$ associated to the small Gobel in $\mathcal{G}_{\mathbb{B}}^2$ is

$$0 \leftarrow \mathbb{B}^1 \xleftarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} \mathbb{B}^2 \xleftarrow{\begin{pmatrix} -f_2 & c_{11} & c_{21} \\ f_1 & c_{12} & c_{22} \end{pmatrix}} \mathbb{B}^3 \xleftarrow{\begin{pmatrix} -c_{12} & -c_{22} & c_{11} & c_{21} \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{pmatrix}} \mathbb{B}^4 \xleftarrow{\begin{pmatrix} -f_2 & 0 & c_{11} & c_{21} & 0 \\ 0 & -f_2 & 0 & c_{11} & c_{21} \\ f_1 & 0 & c_{12} & c_{22} & 0 \\ 0 & f_1 & 0 & c_{12} & c_{22} \end{pmatrix}} \mathbb{B}^5 \leftarrow \dots \quad (5)$$

$$\dots \leftarrow \mathbb{B}^{2j-1} \xleftarrow{\varphi_j} \mathbb{B}^{2j} \xleftarrow{\psi_j} \mathbb{B}^{2j+1} \leftarrow \dots$$

$$\varphi_j = \left(\begin{array}{ccc|ccc} -c_{12} & -c_{22} & & c_{11} & c_{21} & \\ & \ddots & & \ddots & \ddots & \\ & & -c_{12} & -c_{22} & c_{11} & c_{21} \\ \hline & & f_1 & & f_2 & \\ & & & \ddots & & \\ & & & & f_1 & \\ & & & & & f_2 \end{array} \right), \quad \psi_j = \left(\begin{array}{ccc|ccc} -f_2 & & & c_{11} & c_{21} & \\ & \ddots & & \ddots & \ddots & \\ & & -f_2 & & c_{11} & c_{21} \\ \hline & & f_1 & & c_{12} & c_{22} \\ & & & \ddots & \ddots & \\ & & & & f_1 & \\ & & & & & c_{12} & c_{22} \end{array} \right)$$

By choosing a linear map $L : \mathbb{B} \longrightarrow K$ which is non-zero on the 1-dimensional socle on \mathbb{B} , we obtain a non-degenerate bilinear form:

$$\cdot_L : \mathbb{B} \times \mathbb{B} \longrightarrow K \quad a \cdot_L b := L(a \cdot b),$$

where $a \cdot b = ab$ denotes the multiplication of a and b in the ring \mathbb{B} (see [3, Sec.21.2]). We induce also a non-degenerate K -bilinear form on (the finite K -dimensional vector space) \mathbb{B}^2 by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot_L \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} := L(a_1 \cdot b_1 + a_2 \cdot b_2)$$

Introducing the homology (cohomology) module of the Koszul complexes over \mathbb{B} defined by f_1 and f_2 (Section 3.3):

$$H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) := \frac{\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle^\perp}{\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \rangle}, \quad H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*) := \frac{\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \rangle^\perp}{\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle},$$

where $\langle * \rangle$ denotes the \mathbb{B} -module generated by $*$ and $*^\perp$ is the \cdot_L -orthogonal subspace to $*$. The above bilinear form on \mathbb{B} induces non-degenerate bilinear forms on the first (co)homology groups. We have that

$$\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}, \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \in H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) \quad , \quad \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix}, \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} \in H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*)$$

are elements in the first Koszul homology (cohomology) module of f_1, f_2 over \mathbb{B} .

The hyperhomology of the small Gobelins $\mathcal{G}_{\mathbb{B}}^1$ constructed from the first syzygy $c_{11}f_1 + c_{12}f_2 = 0$ over \mathbb{B} is:

Proposition 0.1. *The hyperhomology groups of the small Gobelins $\mathcal{G}_{\mathbb{B}}^1$ are*

$$\mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1) = \frac{\mathbb{B}}{(f_1, f_2)} \quad , \quad \mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^1) = \frac{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}{\langle \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \rangle} \quad (6)$$

and for $j \geq 1$

$$\mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1) = \frac{\langle \begin{pmatrix} -f_2 \\ c_{11} \end{pmatrix}, \begin{pmatrix} f_1 \\ c_{12} \end{pmatrix} \rangle^\perp}{\langle \begin{pmatrix} -c_{12} \\ f_1 \end{pmatrix}, \begin{pmatrix} c_{11} \\ f_2 \end{pmatrix} \rangle} \quad , \quad \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1) = \frac{\langle \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \rangle_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}^\perp}{\langle \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \rangle_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}} \quad (7)$$

where these last 2 groups are equidimensional as K -vector spaces.

The invariants that we will use to describe the hyperhomology of $\mathbb{H}_*(\mathcal{G}_{\mathbb{B}}^2)$ in terms of $\mathbb{H}_*(\mathcal{G}_{\mathbb{B}}^1)$ are two flags of ideals in the ring \mathbb{B} :

$$0 = L_0 \subset L_1 \subset \cdots \subset L_\infty \subset F_\infty \subset \cdots \subset F_1 \subset F_0 = \mathbb{B} \quad (8)$$

and

$$0 = L'_0 \subset L'_1 \subset \cdots \subset L'_\infty \subset F'_\infty \subset \cdots \subset F'_1 \subset F'_0 = \mathbb{B} \quad (9)$$

where the first flag is defined for ascending $j \geq 1$ by:

$$L_j := (L_{j-1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} \quad , \quad F_j := (F_{j-1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} \subset \mathbb{B}$$

and L_∞ and F_∞ are the smallest and largest ideals in \mathbb{B} satisfying $I = (I \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix})$, respectively. The other flag is defined by inverting the roles of the 2 syzygies:

$$L'_j := (L'_{j-1} \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} \quad , \quad F'_j := (F'_{j-1} \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} \subset \mathbb{B}.$$

Our main result is:

Theorem 0.1. *Let \mathbb{B} be a Gorenstein K -algebra of dimension 0, assume given a matrix identity (4), then the hyperhomology groups of the small Gobelins $\mathcal{G}_{\mathbb{B}}^2$ are:*

$$\mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^2) = \frac{\mathbb{B}}{(f_1, f_2)} \quad , \quad \mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^2) = \frac{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}{< \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}, \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} >_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}}$$

and for $j \geq 1$ we have exact sequences:

$$0 \leftarrow \frac{F_{j-1}}{F_{j-1} \cap F'_1} \xleftarrow{\partial} \mathbb{H}_{2j-2}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j}^*} \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j}} \frac{\mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{\text{Ann}_{\mathbb{B}}(L'_j \cap L_1)}{\text{Ann}_{\mathbb{B}}(L_1)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \leftarrow 0 \quad (10)$$

and

$$0 \leftarrow \frac{\text{Ann}_{\mathbb{B}}(L'_j \cap L_1)}{\text{Ann}_{\mathbb{B}}(L_1)} \xleftarrow{\partial} \mathbb{H}_{2j-1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j+1}^*} \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j+1}} \frac{\mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{F_j}{F_j \cap F'_1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}} \leftarrow 0 \quad (11)$$

The invariants coming into the theorem are contained in the intersection of the flag L'_* with L_1 and in the induced flag F_* on $\frac{\mathbb{B}}{F'_1}$. Or interchanging the roles of the 2 syzygies, in the intersection of the flag L_* with L'_1 and in the induced flag F'_* on $\frac{\mathbb{B}}{F_1}$.

These computations are useful for giving formulas for the topological (GSV) and homological indices of a vector field with isolated singularities tangent to a complete intersection with an isolated singularity (see [1, 4, 5]), where φ in equation (2) is the Jacobian Matrix Df of f and (2) is the tangency condition of the vector field along the complete intersection $f = 0$. In the isolated singularity case, the \mathbb{B} algebra is the K -algebra of functions on the zero locus $X = 0$ of the vector field, which is an isolated but multiple point. So the algebra we develop here is inside the zero set of the vector field.

1 The small Gobelins

All tensor products will be over K , unless otherwise specified. Let F, G and H be finite dimensional K -vector spaces of dimensions N, ℓ , and k respectively. Then the equation (2) gives rise to the anticommutative square

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{-f} & H \otimes \mathcal{O} \\ X \downarrow & & \downarrow c \\ F \otimes \mathcal{O} & \xrightarrow{\varphi} & G \otimes \mathcal{O} \end{array} \quad (12)$$

Let $\mathbb{P}^{\ell-1}$ denote the projective space, $\text{Proj}(G)$, and $\mathcal{O}_{\mathbb{P}^{\ell-1}}(1)$ the sheaf of hyperplane sections on $\mathbb{P}^{\ell-1}$. Let s_1, \dots, s_{ℓ} denote a basis of its global sections, $s := (s_1, \dots, s_{\ell})$, $\tilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}$ and $\tilde{\mathcal{O}}(m) := \mathcal{O} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(1)^{\otimes m}$. We tensor the square (12) with the sheaf $\mathcal{O}_{\mathbb{P}^{\ell-1}}$ and continue at the right bottom of the square with the tensor product of the natural morphism

$$s \cdot : G \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{\ell-1}}(1)$$

with \mathcal{O} to obtain the following anticommutative square of $\tilde{\mathcal{O}}$ -sheaves on $\mathbb{P}^{\ell-1}$:

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{-f} & H \otimes \tilde{\mathcal{O}} \\ X \downarrow & & \downarrow s \cdot c \\ F \otimes \tilde{\mathcal{O}} & \xrightarrow{s \cdot \varphi} & \tilde{\mathcal{O}}(1) \end{array}$$

Since going around the square gives a 1×1 matrix, we can transpose the upper part of the square and obtain the anticommutative square

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{(s \cdot c)^t} & H^* \otimes \tilde{\mathcal{O}}(1) \\ X \downarrow & & \downarrow -f^t \\ F \otimes \tilde{\mathcal{O}} & \xrightarrow{s \cdot \varphi} & \tilde{\mathcal{O}}(1) \end{array}. \quad (13)$$

It is explained in [1] how from this data one constructs a double complex, called the Gobelin.

Over the ring $\mathbb{B} := \mathcal{O}/(X_1, \dots, X_N)$ the identity (2) reduces to

$$cf = \begin{pmatrix} c_{11} & \dots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{\ell 1} & \dots & c_{\ell k} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} = 0. \quad (14)$$

Using the notation $\tilde{\mathbb{B}}_{\mathbb{P}^{\ell-1}} := \mathbb{B} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}$ and $\tilde{\mathbb{B}}_{\mathbb{P}^{\ell-1}}(1) := \mathbb{B} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(1)$, (13) gives rise to a syzygy

$$\tilde{\mathbb{B}}_{\mathbb{P}^{\ell-1}} \xrightarrow{(s \cdot c)^t} H^* \otimes \tilde{\mathbb{B}}_{\mathbb{P}^{\ell-1}}(1) \xrightarrow{-f^t} \tilde{\mathbb{B}}_{\mathbb{P}^{\ell-1}}(1).$$

Considering the Koszul complexes formed with each term of the syzygy, tensoring with $\mathcal{O}_{\mathbb{P}^{\ell-1}}(d)$, taking global sections and dualizing we weave these Buchsbaum-Eisenbud complexes (see [3, Pag. 589]) to form a double complex, called the small Gobelin in [1], which we denote by

$$\mathcal{G}_{\mathbb{B}}^{\ell} := \{\mathcal{G}_{\mathbb{B}, i, j}^{\ell} := D_i G^* \otimes \Lambda^{\ell+i-j} H \otimes \mathbb{B}\}$$

where $D_i G^* := H^0(\mathbb{P}^{\ell-1}, \mathcal{O}_{\mathbb{P}^{\ell-1}}(i))^*$ is the homogeneous component of the divided power algebra of $K[s_1, \dots, s_{\ell}]$ of degree i , and the connecting maps are constructed using f for the vertical strands and c for the horizontal ones.

Figure 1: The lower left hand part of the small Gobelin $\mathcal{G}_{\mathbb{B}}^2$ for $k = 2$, $l = 2$ beginning at $(0, 0)$.

2 Morphisms between small Gobelins

Let $r < \ell$. We will modify the identity (14), considering only the first r rows, that is:

$$\begin{pmatrix} c_{11} & \dots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{r1} & \dots & c_{rk} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} = 0. \quad (15)$$

Let $G_r \subset G$ be the vector subspace obtained by setting $s_{r+1} = \dots = s_r = 0$ and denote by $\mathcal{G}_{\mathbb{B}}^r$ the small Gobelin constructed from the syzygy associated to the equation (15)

$$\widetilde{\mathbb{B}}_{\mathbb{P}^{r-1}} \xrightarrow{(s' \cdot c)^t} H^* \otimes \widetilde{\mathbb{B}}_{\mathbb{P}^{r-1}}(1) \xrightarrow{-f^t} \widetilde{\mathbb{B}}_{\mathbb{P}^{r-1}}(1).$$

with $\mathbb{P}^{r-1} := \text{Proj}(G_r) \subset \mathbb{P}^{\ell-1}$ and $s' := (s_1, \dots, s_r)$. Let $D_i G_r^* := H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(i))^*$ be the homogeneous component of the divided power algebra of $K[s_1, \dots, s_r]$ of degree i .

Consider the maps induced by multiplication by s_r :

$$\sigma : H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(i)) \rightarrow H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(i+1)) \quad s^\alpha \rightarrow s^\alpha s_r.$$

Note that

$$H^0(\mathbb{P}^{r-2}, \mathcal{O}_{\mathbb{P}^{r-2}}(i+1)) = \frac{H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(i+1))}{\sigma(H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(i)))}$$

These morphisms induces by duality the morphisms

$$\begin{aligned} \sigma^* : \mathcal{G}_{\mathbb{B}, i, j}^r &= D_i G_r^* \otimes \wedge^{r+i-j} H \otimes \mathbb{B} \rightarrow \mathcal{G}_{\mathbb{B}, i-1, j-1}^r = D_{i-1} G_r^* \otimes \wedge^{r+i-j} H \otimes \mathbb{B} \\ a \otimes \eta \otimes b &\rightarrow \sigma^*(a) \otimes \eta \otimes b. \end{aligned}$$

which consists of contracting with the last variable s_r . Let $\mathcal{G}_{\mathbb{B}}^r(-1, -1)$ denote the shift of $\mathcal{G}_{\mathbb{B}}^r$ by $(-1, -1)$, i.e.

$$\mathcal{G}_{\mathbb{B}}^r(-1, -1)_{i, j} = \mathcal{G}_{\mathbb{B}, i-1, j-1}^r.$$

Hence σ^* defines a morphism between the double complexes $\mathcal{G}_{\mathbb{B}}^r$ and $\mathcal{G}_{\mathbb{B}}^r(-1, -1)$. Note that in figure 1, σ^* corresponds to antidiagonal arrows descending vertically one and moving horizontally to the left by one.

Let $\iota : \mathcal{G}_{\mathbb{B}}^{r-1} \rightarrow \mathcal{G}_{\mathbb{B}}^r$ be the inclusion defined in each term of the double complexes by

$$\iota : \mathcal{G}_{\mathbb{B}, i, j}^{r-1} = D_i G_{r-1}^* \otimes \wedge^{r-1+i-j} H \otimes \mathbb{B} \rightarrow \mathcal{G}_{\mathbb{B}, i, j+1}^r = D_i G_r^* \otimes \wedge^{r+i-j-1} H \otimes \mathbb{B}.$$

Directly from above we obtain

Theorem 2.1. *The following is a short exact sequence of double complexes of \mathbb{B} -modules*

$$0 \longrightarrow \mathcal{G}_{\mathbb{B}}^{r-1} \xrightarrow{\iota} \mathcal{G}_{\mathbb{B}}^r \xrightarrow{\sigma^*} \mathcal{G}_{\mathbb{B}}^r(-1, -1) \longrightarrow 0.$$

giving rise to a long exact sequence of hyperhomology groups

$$\dots \longrightarrow \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^{r-1}) \xrightarrow{\iota} \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^r) \xrightarrow{\sigma^*} \mathbb{H}_{j-2}(\mathcal{G}_{\mathbb{B}}^r) \xrightarrow{\delta} \mathbb{H}_{j-1}(\mathcal{G}_{\mathbb{B}}^{r-1}) \longrightarrow \dots$$

This long exact sequence relates the hyperhomology groups $\mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^r)$ of the small Gobelin $\mathcal{G}_{\mathbb{B}}^r$ via σ^* with $\mathbb{H}_{j-2}(\mathcal{G}_{\mathbb{B}}^r)$ and we find the kernels and cokernels in the homology in the small Gobelin $\mathcal{G}_{\mathbb{B}}^{r-1}$ with $(r-1)$ -syzygies between f_1, \dots, f_k . In particular when $r = 1$, the kernels and cokernels will be 0, giving the isomorphism between even and odd homology groups for positive degrees, as in [5] for the hypersurface case.

3 Some auxiliary algebra

To develop the pattern that follows from Theorem 2.1, we will analyse the case where $k = 2$ and $\ell = 1, 2$. We will assume from now on further that \mathcal{O} is a local Gorenstein K -algebra of dimension N , with K as residue field and that X_1, \dots, X_N is a regular sequence. Hence we have that \mathbb{B} is a local Gorenstein K -algebra of dimension 0 which is a finite dimensional vector space over K , say of dimension μ . Multiplication in \mathbb{B} will be denote by $a \cdot b$, or simply ab .

3.1 The ideals associated to f_1 and f_2 .

Let $f_1, f_2 \in \mathbb{B}$ be non-units. Denote by ν_1, ν_2 and ν the codimension of the ideals $(f_1), (f_2)$ and (f_1, f_2) in \mathbb{B} . It follows from the short exact sequence

$$0 \longrightarrow (f_1) \cap (f_2) \xrightarrow{(id, -id)} (f_1) \oplus (f_2) \xrightarrow{+} (f_1, f_2) \longrightarrow 0$$

that the codimension of $(f_1) \cap (f_2)$ is $\nu_1 + \nu_2 - \nu$ and from the non-degenerate duality that $\text{Ann}_{\mathbb{B}}(f_1) \cap \text{Ann}_{\mathbb{B}}(f_2) = \text{Ann}_{\mathbb{B}}(f_1, f_2)$ has dimension ν , $\text{Ann}_{\mathbb{B}}(f_i)$ has dimension ν_i and $\text{Ann}_{\mathbb{B}}((f_1) \cap (f_2))$ has dimension $\nu_1 + \nu_2 - \nu$:

$$\begin{array}{ccc} \begin{array}{c} \mathbb{B} \\ \cup \\ (f_1, f_2) \\ \cup \\ (f_1) \quad (f_2) \\ \cup \\ (f_1) \cap (f_2) \\ \cup \\ 0 \end{array} & \text{Ann}_{\mathbb{B}}((f_1) \cap (f_2)) = \langle \text{Ann}_{\mathbb{B}}(f_1), \text{Ann}_{\mathbb{B}}(f_2) \rangle \\ & \begin{array}{c} \mathbb{B} \\ \cup \\ \text{Ann}_{\mathbb{B}}(f_1) \\ \cup \\ \text{Ann}_{\mathbb{B}}(f_2) \\ \cup \\ \text{Ann}_{\mathbb{B}}(f_1, f_2) \\ \cup \\ 0 \end{array} \end{array}$$

From the exact sequence

$$0 \longrightarrow \text{Ann}_{\mathbb{B}}(f_1) \rightarrow \mathbb{B} \xrightarrow{f_1} (f_1) \longrightarrow 0$$

we deduce the sequence

$$0 \longrightarrow \text{Ann}_{\mathbb{B}}(f_1) \rightarrow (f_2 : f_1) \xrightarrow{f_1} (f_1) \cap (f_2) \longrightarrow 0$$

so that the dimension of $(f_2 : f_1)$ is $\mu - \nu_2 + \nu$, and similarly the dimension of $(f_1 : f_2)$ is $\mu - \nu_1 + \nu$. The modules

$$\frac{(f_2 : f_1)}{(f_2)}, \quad \frac{(f_1 : f_2)}{(f_1)}$$

have dimension ν . From the exact sequence

$$0 \longrightarrow \text{Ann}_{\mathbb{B}}(f_1) \cap \text{Ann}_{\mathbb{B}}(f_2) \rightarrow \text{Ann}_{\mathbb{B}}(f_2) \xrightarrow{f_1} f_1 \text{Ann}_{\mathbb{B}}(f_2) \longrightarrow 0$$

we see that the dimension of $f_1 \text{Ann}_{\mathbb{B}}(f_2)$ is $\nu_2 - \nu$ and similarly the dimension of $f_2 \text{Ann}_{\mathbb{B}}(f_1)$ is $\nu_1 - \nu$. Hence the modules

$$\frac{\text{Ann}_{\mathbb{B}}(f_2)}{f_1 \text{Ann}_{\mathbb{B}}(f_2)}, \quad \frac{\text{Ann}_{\mathbb{B}}(f_1)}{f_2 \text{Ann}_{\mathbb{B}}(f_1)}$$

have dimension ν .

3.2 Non-degenerate bilinear forms

Let $L : \mathbb{B} \longrightarrow K$ be a trace map, i.e. a functional which is non trivial on the socle of \mathbb{B} (see [2, 6]). The bilinear form on \mathbb{B}

$$\cdot_L : \mathbb{B} \oplus \mathbb{B} \xrightarrow{\cdot} \mathbb{B} \xrightarrow{L} K, \quad a \cdot_L b = L(a \cdot b), \quad (16)$$

is non-degenerate, having the property that if I is an ideal in \mathbb{B} then its orthogonal I^\perp is $\text{Ann}_{\mathbb{B}}(I)$, which is independent of the chosen trace L .

Introduce on \mathbb{B}^r the direct sum non-degenerate symmetric bilinear forms:

$$\cdot : \mathbb{B}^r \oplus \mathbb{B}^r \longrightarrow \mathbb{B}, \quad \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} := \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}^t \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \sum_{i=1}^r a_i b_i, \quad (17)$$

$$\cdot_L : \mathbb{B}^r \oplus \mathbb{B}^r \longrightarrow K, \quad \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \cdot_L \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} := L\left(\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}^t \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}\right) = L\left(\sum_{i=1}^r a_i b_i\right), \quad (18)$$

Given a submodule $M \subset \mathbb{B}^r$, we will denote by $M^\perp \subset \mathbb{B}^r$ its orthogonal. It has complementary dimension, due to the non-degeneracy of the bilinear form (18), and it is also a submodule. If m_1, \dots, m_s are generators of a module M we will denote by $\langle m_1, \dots, m_r \rangle$ the module generated by them and by $\langle m_1, \dots, m_r \rangle^\perp$ its orthogonal. If M_1, M_2 are submodules of \mathbb{B}^r , then $M_1^\perp \cap M_2^\perp = \langle M_1 \cup M_2 \rangle^\perp$.

Note that the involution

$$\kappa : \mathbb{B}^2 \longrightarrow \mathbb{B}^2, \quad \kappa(a, b) := (-b, a), \quad (19)$$

is an \cdot_L -automorphism:

$$\kappa \begin{pmatrix} a \\ b \end{pmatrix} \cdot_L \kappa \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \cdot_L \begin{pmatrix} -d \\ c \end{pmatrix} = L(bd + ac) = \begin{pmatrix} a \\ b \end{pmatrix} \cdot_L \begin{pmatrix} c \\ d \end{pmatrix}.$$

3.3 The Koszul complexes $\mathcal{K}_{\mathbb{B}}(f_1, f_2)$ and $\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*$

Denote by $\mathcal{K}_{\mathbb{B}}(f_1, f_2)$ the Koszul complex built with f_1, f_2 over \mathbb{B} :

$$0 \longleftarrow \mathbb{B} \xleftarrow{(f_1, f_2)} \mathbb{B}^2 \xleftarrow{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}} \mathbb{B} \longleftarrow 0 \quad (20)$$

with homology groups

$$H_0(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) = \frac{\mathbb{B}}{(f_1, f_2)}, H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) = \frac{\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle^\perp}{\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \rangle}, \text{ and } H_2(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) = \text{Ann}_{\mathbb{B}}(f_1, f_2). \quad (21)$$

The dimension of $\frac{\mathbb{B}}{(f_1, f_2)}$ is ν , $H_2(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ also has dimension ν since it is orthogonal to (f_1, f_2) and $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ has dimension 2ν , since the alternating sum of the dimensions of the modules in (20) is 0, and hence also the Euler characteristic of its homology groups.

From the exact sequence

$$0 \longrightarrow \text{Ann}_{\mathbb{B}}(f_1) \cap \text{Ann}_{\mathbb{B}}(f_2) \rightarrow \mathbb{B} \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle \longrightarrow 0$$

we also see that the dimension of $\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle$ is $\mu - \nu$, (similarly of $\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \rangle$) and by orthogonality in \mathbb{B}^2 , the dimension of $\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle^\perp$ and of $\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \rangle^\perp$ is $\mu + \nu$. This gives another proof that $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ has dimension 2ν .

Lemma 3.1. *The inclusions i and projections π of \mathbb{B}^2 to the factors induces the exact sequences:*

$$\begin{aligned} 0 &\longrightarrow \frac{\text{Ann}_{\mathbb{B}}(f_2)}{f_1 \text{Ann}_{\mathbb{B}}(f_2)} \xrightarrow{i_2} H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) \xrightarrow{\pi_1} \frac{(f_2 : f_1)}{(f_2)} \longrightarrow 0 \\ 0 &\longrightarrow \frac{\text{Ann}_{\mathbb{B}}(f_1)}{f_2 \text{Ann}_{\mathbb{B}}(f_1)} \xrightarrow{i_1} H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) \xrightarrow{\pi_2} \frac{(f_1 : f_2)}{(f_1)} \longrightarrow 0 \end{aligned}$$

Proof. We prove the second one. The projection π_2 induces a map $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) \xrightarrow{\pi_2} \frac{\mathbb{B}}{(f_1)}$ due to the boundaries $c \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$. Now $\begin{pmatrix} a \\ b \end{pmatrix} \in H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ means $af_1 + bf_2 = 0$, hence $bf_2 = -af_1$, so $b \in (f_1 : f_2)$. And conversely, given $b \in (f_1 : f_2)$ there is an $a \in \mathbb{B}$ with $bf_2 = -af_1$, and we may form the class $(a, b)^t \in H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$.

An element in $\text{Ker}(\pi_2) \subset H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ may be written as $\begin{pmatrix} a \\ 0 \end{pmatrix} + b \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \in \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^\perp$ that is, $af_1 = 0$, and hence $a \in \text{Ann}_{\mathbb{B}}(f_1)$. The element $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is 0 in $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ if $\begin{pmatrix} a \\ 0 \end{pmatrix} = c \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$, hence $a = -cf_2$ with $cf_1 = 0$; that is, $a \in f_2 \text{Ann}_{\mathbb{B}}(f_1)$. \square

The algorithm to construct an element in $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ is then to begin with an element $b \in (f_1 : f_2) \subset \mathbb{B}$, so it satisfies a relation $bf_2 = -af_1$ for some $a \in \mathbb{B}$, and then form $\begin{pmatrix} a \\ b \end{pmatrix} \in H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$. The a is unique up to adding an element in $\text{Ann}_{\mathbb{B}}(f_1)$. If $b = cf_1$ then $\begin{pmatrix} a \\ b \end{pmatrix} = c \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$ represents the 0 element.

The dual Koszul complex $\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*$ is

$$0 \longrightarrow \mathbb{B} \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \mathbb{B}^{2(-f_2, f_1)} \mathbb{B} \longrightarrow 0$$

with cohomology groups

$$H^0(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*) = \text{Ann}_{\mathbb{B}}(f_1, f_2), \quad H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*) = \frac{\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \rangle^\perp}{\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle}, \quad \text{and} \quad H^2(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*) = \frac{\mathbb{B}}{(f_1, f_2)}.$$

The bilinear form \cdot_L (17) in \mathbb{B}^2 induces non-degenerate bilinear forms in $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ and in $H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*)$, and the \mathbb{B} -isomorphism κ in (19) induces a \mathbb{B} -module isomorphism

$$\kappa^* : H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) \longrightarrow H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*)$$

which preserves the bilinear forms.

4 The hyperhomology of the Gobelins $\mathcal{G}_{\mathbb{B}}^1$.

4.1 The hyperhomology of $\mathcal{G}_{\mathbb{B}}^1$.

The small Gobelins $\mathcal{G}_{\mathbb{B}}^1$ constructed from

$$(c_{11} \quad c_{12}) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \tag{22}$$

is given by Figure 1 with $G = G_1$. The total complex $\mathcal{G}_{\mathbb{B}}^1$ has the form

$$0 \longleftarrow \mathbb{B} \xleftarrow{(f_1, f_2)} \mathbb{B}^2 \xleftarrow{C_\psi} \mathbb{B}^2 \xleftarrow{C_\varphi} \mathbb{B}^2 \xleftarrow{C_\psi} \mathbb{B}^2 \xleftarrow{C_\varphi} \dots \quad (23)$$

where

$$C_\psi = \begin{pmatrix} -f_2 & c_{11} \\ f_1 & c_{12} \end{pmatrix}, \quad C_\varphi = \begin{pmatrix} -c_{12} & c_{11} \\ f_1 & f_2 \end{pmatrix}.$$

We have $(\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}) \in \langle \begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix} \rangle^\perp$, i.e. it satisfies a relation in \mathbb{B} of the form $c_{11}f_1 + c_{12}f_2 = 0$.

Proof of Proposition 0.1. The computation of $\mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1)$ is direct from the complex (23), as also

$$\mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^1) = \frac{\langle \begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix} \rangle^\perp}{\langle \begin{smallmatrix} -f_2 \\ f_1 \end{smallmatrix} \rangle, (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix})} = \frac{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}{\langle (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}) \rangle}$$

The 2-periodicity of $\mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^1)$ follows from the 2-periodicity of (23). One computes directly

$$\frac{\text{Ker}(C_\psi)}{\text{Im}(C_\varphi)}, \quad \text{and} \quad \frac{\text{Ker}(C_\varphi)}{\text{Im}(C_\psi)}$$

to obtain (7).

We also have for $i, j \geq 1$

$$\begin{aligned} \dim(\mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1)) &= \dim(\text{Ker}(C_\psi)) - \dim(\text{Im}(C_\varphi)) = \dim(\mathbb{B}^2) - \dim(\text{Im}(C_\psi)) - \dim(\text{Im}(C_\varphi)) \\ &= \dim(\text{Ker}(C_\varphi)) - \dim(\text{Im}(C_\psi)) = \dim(\mathbb{H}_{2i+1}(\mathcal{G}_{\mathbb{B}}^1)). \end{aligned}$$

Thus for $j \geq 2$ all the homology groups $\mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^1)$ are equidimensional as K -vector spaces. □

Lemma 4.1. *For $j \geq 1$ the projection map to the second factor*

$$\pi_2 : \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1) \longrightarrow \frac{\mathbb{B}}{(f_1, f_2)}, \quad \pi_2(a, b) = b, \quad (24)$$

induce an exact sequence:

$$0 \longrightarrow \frac{\text{Ann}_{\mathbb{B}}(f_1, f_2)}{\begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \cdot \frac{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}{\langle (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}) \rangle}} \xrightarrow{i_1} \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1) \xrightarrow{\pi_2} \frac{(0 : (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}))_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}}{(f_1, f_2)} \longrightarrow 0 \quad (25)$$

Proof. $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1)$ satisfies $-af_2 + bc_{11} = 0$ and $af_1 + bc_{12} = 0$, or in matrix notation

$$a \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} + b \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = 0 \quad \text{so that} \quad b \in \left(\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \right),$$

and if its cohomology class is 0 then $\begin{pmatrix} a \\ b \end{pmatrix} \in \langle \begin{pmatrix} -c_{12} \\ f_1 \end{pmatrix}, \begin{pmatrix} c_{11} \\ f_2 \end{pmatrix} \rangle$ so that $b \in (f_1, f_2)$. This shows that the map π_2 is well defined and surjective.

The elements in the kernel of π_2 may be represented in the form $(a', \alpha_1 f_1 + \alpha_2 f_2)^t$, so that it has a homologous element of the form

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a' \\ \alpha_1 f_1 + \alpha_2 f_2 \end{pmatrix} - \alpha_1 \begin{pmatrix} -c_{12} \\ f_1 \end{pmatrix} - \alpha_2 \begin{pmatrix} c_{11} \\ f_2 \end{pmatrix}.$$

The element $(a, 0)^t$ belongs to $\mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1)$ if and only if $a \in \text{Ann}_{\mathbb{B}}(f_1, f_2)$. It is the 0 element if and only if

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} -c_{12} \\ f_1 \end{pmatrix} + \beta \begin{pmatrix} c_{11} \\ f_2 \end{pmatrix} = \begin{pmatrix} (\alpha, \beta)^t \cdot (-c_{12}, c_{11})^t \\ (\alpha, \beta)^t \cdot (f_1, f_2)^t \end{pmatrix}$$

Hence $(\alpha, \beta)^t \in \langle (f_1, f_2)^t \rangle^{\perp}$. Consider the map

$$\cdot \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} : \langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle^{\perp} \longrightarrow \mathbb{B}$$

which vanishes on $\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}, \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \rangle$. Its image lies in $\text{Ann}_{\mathbb{B}}(f_1, f_2)$ and consists of those elements which represent 0 in $\text{Ker}(i_1)$. \square

4.2 The hypercohomology of $\mathcal{G}_{\mathbb{B}}^{1*}$.

Let $\mathcal{G}_{\mathbb{B}}^{1*}$ be the double complex $\text{Hom}_{\mathbb{B}}(\mathcal{G}_{\mathbb{B}}^1, \mathbb{B})$. The total complex $\text{tot}(\mathcal{G}_{\mathbb{B}}^{1*})$ is obtained by applying the functor $\text{Hom}_{\mathbb{B}}(*, \mathbb{B})$ to (23):

$$0 \longrightarrow \mathbb{B} \xrightarrow{(f_1, f_2)^t} \mathbb{B}^2 \xrightarrow{C_{\psi}^t} \mathbb{B}^2 \xrightarrow{C_{\varphi}^t} \mathbb{B}^2 \xrightarrow{C_{\psi}^t} \mathbb{B}^2 \xrightarrow{C_{\varphi}^t} \dots$$

Lemma 4.2. *The hypercohomology groups of $\mathcal{G}_{\mathbb{B}}^{1*}$ are*

$$\mathbb{H}^0(\mathcal{G}_{\mathbb{B}}^{1*}) = \text{Ann}_{\mathbb{B}}(f_1, f_2), \quad \mathbb{H}^1(\mathcal{G}_{\mathbb{B}}^{1*}) = \langle \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \rangle_{H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}^{\perp}$$

and for $j \geq 1$

$$\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*}) = \frac{\langle \begin{pmatrix} -c_{12} \\ f_1 \end{pmatrix}, \begin{pmatrix} c_{11} \\ f_2 \end{pmatrix} \rangle^{\perp}}{\langle \begin{pmatrix} -f_2 \\ c_{11} \end{pmatrix}, \begin{pmatrix} f_1 \\ c_{12} \end{pmatrix} \rangle}, \quad \mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^{1*}) = \frac{\langle \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \rangle_{H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}^{\perp}}{\langle \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \rangle_{H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}}$$

where these last 2 groups are equidimensional as K -vector spaces.

Proof. Similar as the proof of Proposition 0.1. \square

Lemma 4.3. *For $j \geq 1$ the projection map to the first factor*

$$\pi_1 : \mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*}) \longrightarrow \frac{\mathbb{B}}{(f_1, f_2)}, \quad \pi_1(a, b) = a, \quad (26)$$

induces an exact sequence:

$$0 \longrightarrow \frac{\text{Ann}_{\mathbb{B}}(f_1, f_2)}{\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \cdot \frac{H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*)}{\langle \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \rangle}} \xrightarrow{i_2} \mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*}) \xrightarrow{\pi_1} \frac{(0 : \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix})_{H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*)}}{(f_1, f_2)} \longrightarrow 0 \quad (27)$$

Proof. $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*})$ satisfies $-ac_{12} + bf_1 = 0$ and $ac_{11} + bf_2 = 0$, or in matrix notation

$$a \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} + b \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \quad \text{so that} \quad a \in \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \right),$$

and if its cohomology class is 0 then $\begin{pmatrix} a \\ b \end{pmatrix} \in < \begin{pmatrix} -f_2 \\ c_{11} \end{pmatrix}, \begin{pmatrix} f_1 \\ c_{12} \end{pmatrix} >$ so that $a \in (f_1, f_2)$. This shows that the map π_1 is well defined and surjective.

The elements in the kernel of π_1 may be represented in the form $(\alpha_1 f_1 + \alpha_2 f_2, b')^t$, so that it has a cohomologous element of the form

$$\begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} \alpha_1 f_1 + \alpha_2 f_2 \\ b' \end{pmatrix} - \alpha_1 \begin{pmatrix} f_1 \\ c_{12} \end{pmatrix} + \alpha_2 \begin{pmatrix} -f_2 \\ c_{11} \end{pmatrix}.$$

The element $(0, b)^t$ belongs to $\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*})$ if and only if $b \in \text{Ann}_{\mathbb{B}}(f_1, f_2)$. It is the 0 element if and only if

$$\begin{pmatrix} 0 \\ b \end{pmatrix} = \alpha \begin{pmatrix} -f_2 \\ c_{11} \end{pmatrix} + \beta \begin{pmatrix} f_1 \\ c_{12} \end{pmatrix} = \begin{pmatrix} (\alpha, \beta)^t \cdot (-f_2, f_1)^t \\ (\alpha, \beta)^t \cdot (c_{11}, c_{12})^t \end{pmatrix}$$

Hence $(\alpha, \beta)^t \in < (-f_2, f_1)^t >^\perp$. Consider the map

$$\cdot \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} : < \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} >^\perp \longrightarrow \mathbb{B}$$

which vanishes on $< \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} >$. Its image lies in $\text{Ann}_{\mathbb{B}}(f_1, f_2)$ and consists of those elements which represent 0 in $\text{Ker}(i_2)$. \square

5 The hyperhomology of the Gobelins $\mathcal{G}_{\mathbb{B}}^2$.

5.1 The complex $\text{tot}(\mathcal{G}_{\mathbb{B}}^2)$.

The small Gobelins $\mathcal{G}_{\mathbb{B}}^2$ constructed from

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \quad (28)$$

is given by Figure 1 with $G = G_2$. The total complex $\text{tot}(\mathcal{G}_{\mathbb{B}}^2)$ is

$$0 \leftarrow \mathbb{B}^1 \xleftarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} \mathbb{B}^2 \xleftarrow{\begin{pmatrix} -f_2 & c_{11} & c_{21} \\ f_1 & c_{12} & c_{22} \end{pmatrix}} \mathbb{B}^3 \xleftarrow{\begin{pmatrix} -c_{12} & -c_{22} & c_{11} & c_{21} \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{pmatrix}} \mathbb{B}^4 \xleftarrow{\begin{pmatrix} -f_2 & 0 & c_{11} & c_{21} & 0 \\ 0 & -f_2 & 0 & c_{11} & c_{21} \\ f_1 & 0 & c_{12} & c_{22} & 0 \\ 0 & f_1 & 0 & c_{12} & c_{22} \end{pmatrix}} \mathbb{B}^5 \leftarrow \dots \quad (29)$$

$$\dots \leftarrow \mathbb{B}^{2j-1} \xleftarrow{\varphi_j} \mathbb{B}^{2j} \xleftarrow{\psi_j} \mathbb{B}^{2j+1} \xleftarrow{\varphi_j} \mathbb{B}^{2j+2} \leftarrow \dots \quad (30)$$

$$\varphi_j = \left(\begin{array}{cc|cc} -c_{12} & -c_{22} & c_{11} & c_{21} \\ & \ddots & \ddots & \ddots \\ & & -c_{12} & -c_{22} \\ \hline f_1 & & f_2 & \\ & \ddots & \ddots & \\ & & f_1 & f_2 \end{array} \right), \quad \psi_j = \left(\begin{array}{cc|cc} -f_2 & & c_{11} & c_{21} \\ & \ddots & \ddots & \ddots \\ & & -f_2 & \\ \hline f_1 & & c_{12} & c_{22} \\ & \ddots & \ddots & \\ & & f_1 & \end{array} \right)$$

5.2 Even dimensional hyperhomology $\mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^2)$

Let $\alpha := (a_1, \dots, a_j, b_1, \dots, b_{j+1})^t \in \mathbb{B}^{2j+1}$ be a cycle $\psi_j(\alpha) = 0$, for $j \geq 1$. The cycle condition consists of $2j$ equalities, and they may be organized into j pairs of equations by considering the i and $i + j$ terms to obtain for $i = 1, \dots, j$:

$$a_i \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} + b_i \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} + b_{i+1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (31)$$

Considering these equations in $H_1(K_{\mathbb{B}}(f_1, f_2))$ they become

$$b_i \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} + b_{i+1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in H_1(K_{\mathbb{B}}(f_1, f_2)) \quad (32)$$

This means that in order to solve (31) we first solve (32), to obtain (b_1, \dots, b_{j+1}) , and then afterwards choose the (a_1, \dots, a_j) so that (31) is satisfied. Such a_i always exist and are unique up to adding an element in $\text{Ann}_{\mathbb{B}}(f_1, f_2)$. This and a direct inspection of the columns of φ_j give that the projection $\rho_j(\alpha) = (b_1, \dots, b_{j+1})$ induces the horizontally exact diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{B}^j & \rightarrow & \mathbb{B}^{2j+1} & \xrightarrow{\rho_j} & \mathbb{B}^{j+1} \rightarrow 0 \\ & & \cup & & \cup & & \cup \\ 0 & \rightarrow & \text{Ann}_{\mathbb{B}}(f_1, f_2)^j & \rightarrow & \text{Ker}(\psi_j) & \rightarrow & \rho_j(\text{Ker}(\psi_j)) \rightarrow 0 \\ & & \cup & & \cup & & \cup \\ 0 & \rightarrow & \left(\begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^\perp + \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^\perp \right)^j & \rightarrow & \text{Im}(\varphi_j) & \rightarrow & (f_1, f_2)^{j+1} \rightarrow 0 \end{array}$$

In particular, we get a well defined map

$$\tilde{\rho}_j : \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^2) \rightarrow \left(\frac{\mathbb{B}}{(f_1, f_2)} \right)^{j+1},$$

which tells us that the b_j components of the cycle α are invariants of the hyperhomology classes, when considered in the ring $\frac{\mathbb{B}}{(f_1, f_2)}$. Recall the definition of the flag of ideals (8) and (9) in \mathbb{B} .

Proposition 5.1. *Let $\alpha := (a_1, \dots, a_j, b_1, \dots, b_{j+1})^t \in \mathbb{B}^{2j+1}$ be a cycle $\psi_j(\alpha) = 0$ with $j \geq 1$, then:*

- a) *For $k = 1, \dots, j + 1$ we have $b_k \in F'_{k-1} \cap F_{j-k+1}$; in particular $b_1 \in F_j$ and $b_{j+1} \in F'_j$.*
- b) *Given $b_1 \in F_j$ we may construct a cycle $\alpha \in \mathbb{B}^{2j+1}$ such that its $j + 1$ component is b_1 .*
- c) *Given $b_{j+1} \in F'_j$ we may construct a cycle $\alpha \in \mathbb{B}^{2j+1}$ such that its $2j + 1$ component is b_{j+1} .*
- d) *The correspondence $b_1 \leftrightarrow b_{j+1}$ established between the components of the j -cycles induces isomorphisms*

$$\Phi_j : \frac{F_j}{L_j} \longrightarrow \frac{F'_j}{L'_j}$$

Proof. a) We have $b_1 \in \mathbb{B} = F'_0$. Equation (32) with $i = 1$ gives $b_2 \in ((\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}) : (\begin{smallmatrix} c_{21} \\ c_{22} \end{smallmatrix}))_{H_1(K_{\mathbb{B}}(f_1, f_2))} = F'_1$. Now Equation (32) with $i = 2$ gives $b_3 \in (F'_1 (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}) : (\begin{smallmatrix} c_{21} \\ c_{22} \end{smallmatrix}))_{H_1(K_{\mathbb{B}}(f_1, f_2))} = F'_2$, since we already know that $b_2 \in F'_1$. Repeating the procedure for increasing i till we arrive for $i = j$ to the conclusion $b_{j+1} \in F'_{j-1}$.

Now beginning at the other extreme, we have $b_{j+1} \in \mathbb{B} = F_0$. Equation (32) with $i = j$ gives $b_j \in ((\begin{smallmatrix} c_{21} \\ c_{22} \end{smallmatrix}) : (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}))_{H_1(K_{\mathbb{B}}(f_1, f_2))} = F_1$. Now Equation (32) with $i = j - 1$ gives $b_{j-1} \in (F_1 (\begin{smallmatrix} c_{21} \\ c_{22} \end{smallmatrix}) : (\begin{smallmatrix} c_{11} \\ c_{12} \end{smallmatrix}))_{H_1(K_{\mathbb{B}}(f_1, f_2))} = F_2$, since we already know that $b_j \in F_1$. Repeating the procedure for increasing i till we arrive for $i = 1$ to the conclusion $b_1 \in F_j$.

b) Now assume that one begins with $b_1 \in F_j$, then one obtains from the hypothesis a $b_2 \in F_{j-1}$ satisfying the equation (32) with $i = 1$. Now using the definition of F_{j-1} one obtains a $b_3 \in$

F_{j-2} satisfying the correspondign equation (32), and so on. Now transporting these equations in $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ to \mathbb{B}^2 one obtains the corresponding equations (31) and the elements $a_1, \dots, a_j \in \mathbb{B}$. Grouping them together into a vector α the equations consist exactly of the cycle condition $\psi_j(\alpha) = 0$.

c) Similar to b).

d) Suppose that we have a cycle that has the form $(0, b_2, \dots, b_{j+1})$. Equation (32) with $i = 1$ gives $b_2 \in (0 : \binom{c_{21}}{c_{22}}) = L'_1$. Equation (32) with $i = 2$ gives $b_3 \in L'_2$ since we already have information on b_2 , till one obtains that $b_{j+1} \in L'_j$. This shows that the map Φ_j is well defined. But by reversing the above procedure, we obtain that Φ_j^{-1} is also well defined; hence Φ_j is an isomorphism \square

Φ_1 gives the isomorphism

$$\frac{\left(\binom{c_{21}}{c_{22}} : \binom{c_{11}}{c_{12}}\right)}{(0 : \binom{c_{11}}{c_{12}})} \leftrightarrow \frac{\left(\binom{c_{11}}{c_{12}} : \binom{c_{21}}{c_{22}}\right)}{(0 : \binom{c_{21}}{c_{22}})}$$

that can be interpreted as being induce from the quotient of the 2 syzygies. Similarly, the other isomorphisms Φ_j are induced from powers of this quotient.

5.3 Even dimensional hypercohomology $\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{2*})$.

Consider the complex $\text{tot}(\mathcal{G}_{\mathbb{B}}^{2*})$ obtained from (29) by applying the functor $\text{Hom}_{\mathbb{B}}(*, \mathbb{B})$. Since \mathbb{B} is a Gorenstein K -algebra of dimension 0 and the complex is by free \mathbb{B} -modules, the dual complex has the form

$$0 \rightarrow \mathbb{B}^1 \xrightarrow{(f_1 \ f_2)^t} \mathbb{B}^2 \xrightarrow{\begin{pmatrix} -f_2 & c_{11} & c_{21} \\ f_1 & c_{12} & c_{22} \end{pmatrix}^t} \mathbb{B}^3 \xrightarrow{\varphi_2^t} \dots \xrightarrow{\varphi_j^t} \mathbb{B}^{2j} \xrightarrow{\psi_j^t} \mathbb{B}^{2j+1} \xrightarrow{\varphi_{j+1}^t} \mathbb{B}^{2j+2} \rightarrow \dots$$

Proposition 5.2. *Let $\beta := (a_1, \dots, a_j, b_1, \dots, b_{j+1})^t \in \mathbb{B}^{2j+1}$, $j \geq 1$, be a cocycle $\varphi_{j+1}^t(\beta) = 0$, then:*

- a) *For $i = 1, \dots, j$ we have $a_i \in L_i \cap L'_{j+1-i}$; in particular $a_1 \in L_1 \cap L'_j$ and $a_j \in L_j \cap L'_1$.*
- b) *Given $a_1 \in L_1 \cap L'_j$, we may construct a cocycle $\beta \in \mathbb{B}^{2j+1}$ such that its first component is a_1 .*
- c) *Given $a_j \in L_j \cap L'_1$, we may construct a cocycle $\beta \in \mathbb{B}^{2j+1}$ such that its j component is a_j .*
- d) *The correspondence $a_1 \leftrightarrow a_j$ established between the components of the j -cocycles induces isomorphisms*

$$\Psi_j : \frac{L_1 \cap L'_j}{L_1 \cap L'_{j-1}} \longrightarrow \frac{L_j \cap L'_1}{L_{j-1} \cap L'_1}$$

Proof. a) The cocycle condition, which consists of $2j + 2$ equalities, may be organized into $j + 1$ pair of equations by considering the i and $i + j$ terms to obtain:

$$a_1 \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} + b_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad a_j \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} + b_{j+1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and for $i = 1, \dots, j - 1$:

$$a_i \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} + a_{i+1} \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} + b_{i+1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Considering these equations in

$$H^1(K_{\mathbb{B}}(f_1, f_2)^*) = \frac{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}^\perp}{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}$$

they become for $i = 1, \dots, j - 1$:

$$a_1 \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad a_i \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} + a_{i+1} \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad a_j \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (33)$$

We note that the \mathbb{B} -involution $\kappa : \mathbb{B}^2 \longrightarrow \mathbb{B}^2$ in (19) has the property that for $\tau, \tau' \in \mathbb{B}^2$ and any ideal $I \subset \mathbb{B}$ we have $(I\tau : \tau') = (I\kappa(\tau) : \kappa(\tau'))$ and hence the flags constructed with the equation (4) coincide with the flags constructed with the equation

$$\begin{pmatrix} -c_{12} & c_{11} \\ -c_{22} & c_{21} \end{pmatrix} \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (34)$$

The cocycle conditions (33) are then:

$$a_1 \in \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} : \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \right) = L_1 \quad , \quad a_j \in \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} : \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} \right) = L'_1$$

and if we begin to solve (*) with $i = 1$ till $i = j - 1$:

$$a_{i+1} \in \left(L_i \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} : \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \right) = L_{i+1}$$

and if we begin to solve (*) with $i = j - 1$ till $i = 1$:

$$a_i \in \left(L'_{j-i} \begin{pmatrix} -c_{12} \\ c_{22} \end{pmatrix} : \begin{pmatrix} -c_{22} \\ c_{21} \end{pmatrix} \right) = L'_{j+1-i}$$

b) and c) Similar to the proofs of b) and c) in Proposition 5.1.

d) If we have 2 j -cocycles with the same first component a_1 , by subtracting them, we obtain a $j - 1$ -cocycle, whose last component will be in $L_j \cap L'_j$, hence $\Psi_j(a_1)$ is well defined modulo $L_j \cap L'_j$. Reversing the argument by beginning with the last component a_j , we obtain Ψ^{-1} . \square

5.4 The long exact sequence of hyperhomology of the small Gobelin

In this case the exact sequence of complexes in Theorem 2.1 is Figure 2, where

$$\begin{array}{cccccccccccccccccccc} & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \leftarrow & \mathbb{B} & \xleftarrow{(f_1, f_2)} & \mathbb{B}^2 & \xleftarrow{C_\psi} & \mathbb{B}^2 & \xleftarrow{C_\varphi} & \mathbb{B}^2 & \xleftarrow{C_\psi} & \mathbb{B}^2 & \xleftarrow{C_\varphi} & \dots & \xleftarrow{C_\varphi} & \mathbb{B}^2 & \xleftarrow{C_\psi} & \mathbb{B}^2 & \xleftarrow{C_\varphi} & \dots \\ & \downarrow \iota_0 & & \downarrow \iota_1 & & \downarrow \iota_2 & & \downarrow \iota_3 & & \downarrow \iota_4 & & & & \downarrow \iota_{2j-1} & & \downarrow \iota_{2j} & & & \\ 0 \leftarrow & \mathbb{B} & \xleftarrow{(f_1, f_2)} & \mathbb{B}^2 & \xleftarrow{\psi_1} & \mathbb{B}^3 & \xleftarrow{\varphi_1} & \mathbb{B}^4 & \xleftarrow{\psi_2} & \mathbb{B}^5 & \xleftarrow{\varphi_2} & \dots & \xleftarrow{\varphi_j} & \mathbb{B}^{2j} & \xleftarrow{\psi_j} & \mathbb{B}^{2j+1} & \xleftarrow{\varphi_{j+1}} & \dots \\ & \downarrow & & \downarrow & & \downarrow \sigma_2^* & & \downarrow \sigma_3^* & & \downarrow \sigma_4^* & & & & \downarrow \sigma_{2j-1}^* & & \downarrow \sigma_{2j}^* & & & \\ 0 \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \mathbb{B} & \xleftarrow{(f_1, f_2)} & \mathbb{B}^2 & \xleftarrow{\psi_1} & \mathbb{B}^3 & \xleftarrow{\varphi_2} & \dots & \xleftarrow{\varphi_{j-1}} & \mathbb{B}^{2j-2} & \xleftarrow{\psi_{j-1}} & \mathbb{B}^{2j-1} & \xleftarrow{\varphi_j} & \dots \\ & & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & \end{array}$$

Figure 2: The short exact sequence of complexes.

$$\iota_{2j} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0_{j-1} \\ b \\ 0_j \end{pmatrix} \in \mathbb{B}^{2j+1}, \quad \iota_{2j+1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0_j \\ b \\ 0_j \end{pmatrix} \in \mathbb{B}^{2j+2}, \quad \sigma_j^* \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ b_1 \\ b_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}$$

where 0_j denotes the 0 column vector of size j , then the boundary maps $\partial_k : \mathbb{H}_{k-1}(\mathcal{G}_{\mathbb{B}}^2) \rightarrow \mathbb{H}_k(\mathcal{G}_{\mathbb{B}}^1)$ are induced from the maps:

$$\partial_{2j+1} : \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ b_1 \\ \vdots \\ b_{j+1} \end{pmatrix} \xrightarrow{\sigma_{2j+1}^*{}^{-1}} \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_j \\ 0 \\ b_1 \\ \vdots \\ b_{j+1} \end{pmatrix} \xrightarrow{\psi_{j+1}} b_1 \begin{pmatrix} c_{21} \\ 0_j \\ c_{22} \\ 0_j \end{pmatrix} \xrightarrow{\iota_{2j}^{-1}} b_1 \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \in \mathbb{B}^2 \quad (35)$$

$$\partial_{2j} : \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ b_1 \\ \vdots \\ b_j \end{pmatrix} \xrightarrow{\sigma_{2j}^*{}^{-1}} \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_j \\ 0 \\ b_1 \\ \vdots \\ b_j \end{pmatrix} \xrightarrow{\varphi_{j+1}} \begin{pmatrix} -c_{22}a_1 + c_{21}b_1 \\ 0_{2j} \end{pmatrix} \xrightarrow{\iota_{2j}^{-1}} \begin{pmatrix} (-c_{22}) \\ c_{21} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \mathbb{B}^2 \quad (36)$$

The long exact sequence of hyperhomology groups of the exact sequence of double complexes in Figure 2 is:

$$\begin{aligned} 0 \leftarrow \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_0} \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1) \xleftarrow{\partial_0} 0 \xleftarrow{\sigma_1^*} \mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_1} \mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^1) \xleftarrow{\partial_1} \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_2^*} \dots \\ \xleftarrow{\sigma_{2j-1}^*} \mathbb{H}_{2j-1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j-1}} \mathbb{H}_{2j-1}(\mathcal{G}_{\mathbb{B}}^1) \xleftarrow{\partial_{2j-1}} \mathbb{H}_{2j-2}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j}^*} \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j}} \end{aligned} \quad (37)$$

$$\xleftarrow{\sigma_{2j+1}^*} \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j+1}} \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1) \xleftarrow{\partial_{2j+1}} \dots \quad (38)$$

For $j \geq 1$, the map

$$\partial_{2j+1} : \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^2) \longrightarrow \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1) = \frac{\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} \rangle^\perp}{\langle \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}, \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \rangle}, \quad \partial_{2j+1} \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ b_1 \\ \vdots \\ b_{j+1} \end{pmatrix} = b_1 \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}$$

has image $F_j \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}$ by Proposition 5.1. The map

$$\begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : F_j \longrightarrow \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1)$$

has kernel $F_j \cap F_1'$ and hence the map $\begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \frac{F_j}{F_j \cap F_1'} \longrightarrow \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1)$ is injective. Hence both maps have the same image and we obtain:

Lemma 5.1. *For $j \geq 1$ the image of ∂_{2j+1} is isomorphic to $\frac{F_j}{F_j \cap F_1'}$.*

Proposition 5.3. *For $j \geq 1$ we have an exact sequence:*

$$\begin{aligned} 0 \leftarrow \frac{F_{j-1}}{F_{j-1} \cap F_1'} \leftarrow \mathbb{H}_{2j-2}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j}^*} \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j}} \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1) \xleftarrow{\partial_{2j}} \\ \xleftarrow{\partial_{2j}} \mathbb{H}_{2j-1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j+1}^*} \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j+1}} \frac{\mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{F_j}{F_j \cap F_1'} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}} \leftarrow 0 \end{aligned} \quad (39)$$

Proof. We are incorporating into the long exact sequence (37) and (38) the conclusion of Lemma 5.1. On the righthand side we are putting $\text{coker}(\partial_{2j+1})$ and on the lefthand side we are putting $\text{Im}(\partial_{2j-1})$. \square

5.5 The long exact sequence of hypercohomology of the dual small Gobelien

Consider the short exact sequence of complexes obtained by applying the functor $\text{Hom}_{\mathbb{B}}(*, \mathbb{B})$ and the long exact sequence of hypercohomology groups. Since \mathbb{B} is a Gorenstein algebra of dimension 0, this long exact sequence is the dual of the sequence (37) and (38) and in particular it contains the dual of (39):

$$\begin{aligned} 0 \rightarrow \left[\frac{F_{j-1}}{F_{j-1} \cap F'_1} \right]^* &\rightarrow \mathbb{H}^{2j-2}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{\sigma_{2j}} \mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{i_{2j}^*} \mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*}) \xrightarrow{\partial_{2j}^*} \\ &\xrightarrow{\partial_{2j}^*} \mathbb{H}^{2j-1}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{\sigma_{2j+1}} \mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{i_{2j+1}^*} \left[\frac{\mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{F_j}{F_j \cap F'_1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}} \right]^* \rightarrow 0 \end{aligned} \quad (40)$$

We want to compute

$$\text{Im}(\partial_{2j}^*) = \text{Ker}(\sigma_{2j+1}) \simeq \text{coker}(i_{2j}^*)$$

Lemma 5.2. *For $j \geq 1$ we have*

$$\text{Im}(i_{2j}^*) = \pi_1^{-1}(L_j \cap L'_1)$$

where π_1 is the projection to the first factor in (27) and L'_1 and L_j are defined in (8) and (9).

Proof. We have $i_{2j}^*(a_1, \dots, b_1, \dots, b_{j+1})^t = (a_1, b_1)^t$. By Proposition 5.2 the only condition on a_1 so that it is part of a cocycle is $a_1 \in L_j \cap L'_1$, and since the only condition on b_1 :

$$a_1 \begin{pmatrix} -c_{12} \\ c_{11} \end{pmatrix} + b_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the same in $\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{2*})$ and in $\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*})$, we have that all admissible b_1 are allowed, so the lemma follows from Proposition 4.3. \square

5.6 Proof of Theorem 0.1

Proof. Now that we have computed $\text{Im}(i_{2j}^*)$, we may split the exact sequence (40) into 2 exact sequences:

$$\begin{aligned} 0 \rightarrow \left[\frac{F_{j-1}}{F_{j-1} \cap F'_1} \right]^* &\rightarrow \mathbb{H}^{2j-2}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{\sigma_{2j}} \mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{i_{2j}^*} \pi_1^{-1}(L_j \cap L'_1) \rightarrow 0 \\ 0 \rightarrow \frac{\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*})}{\pi_1^{-1}(L_j \cap L'_1)} &\xrightarrow{\partial_{2j}^*} \mathbb{H}^{2j-1}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{\sigma_{2j+1}} \mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{i_{2j+1}^*} \left[\frac{\mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{F_j}{F_j \cap F'_1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}} \right]^* \rightarrow 0 \end{aligned} \quad (41)$$

Consider the first term in (41). On using the definition of L'_1 in (9) and Lemma 4.3, the projection π_1 to the first component (26) gives an exact sequence:

$$\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*}) \xrightarrow{\pi_1} L'_1 \rightarrow 0$$

and hence it induces an isomorphism

$$\frac{\mathbb{H}^{2j}(\mathcal{G}_{\mathbb{B}}^{1*})}{\pi_1^{-1}(L_j \cap L'_1)} \xrightarrow{\pi_1} \frac{L'_1}{L_j \cap L'_1}$$

We may then write the exact sequence (41) as:

$$0 \rightarrow \frac{L'_1}{L_j \cap L'_1} \rightarrow \mathbb{H}^{2j-1}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{\sigma_{2j+1}} \mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^{2*}) \xrightarrow{i_{2j+1}^*} \left[\frac{\mathbb{H}^{2j+1}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{F_j}{F_j \cap F'_1}(\mathcal{C}_{22}^{21})} \right]^* \rightarrow 0 \quad (42)$$

Taking duals of the exact sequence

$$0 \longrightarrow L_j \cap L'_1 \longrightarrow L'_1 \longrightarrow \frac{L'_1}{L_j \cap L'_1} \longrightarrow 0$$

and using the non-degenerate bilinear pairing (16) of the Gorenstein algebra \mathbb{B} , we obtain the exact sequence

$$0 \longleftarrow \frac{\mathbb{B}}{\text{Ann}_{\mathbb{B}}(L_j \cap L'_1)} \longleftarrow \frac{\mathbb{B}}{\text{Ann}_{\mathbb{B}}(L'_1)} \longleftarrow \left[\frac{L'_1}{L_j \cap L'_1} \right]^* \longleftarrow 0$$

hence

$$\left[\frac{L'_1}{L_j \cap L'_1} \right]^* = \frac{\text{Ann}_{\mathbb{B}}(L_j \cap L'_1)}{\text{Ann}_{\mathbb{B}}(L'_1)}.$$

Taking duals of the sequence (42), we obtain:

$$0 \longleftarrow \frac{\text{Ann}_{\mathbb{B}}(L_j \cap L'_1)}{\text{Ann}_{\mathbb{B}}(L'_1)} \longleftarrow \mathbb{H}_{2j-1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j+1}^*} \mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{i_{2j+1}} \frac{\mathbb{H}_{2j+1}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{F_j}{F_j \cap F'_1}(\mathcal{C}_{22}^{21})} \longleftarrow 0 \quad (43)$$

Using the exactness of the sequences (39) and (43), we obtain the exact sequence

$$0 \longleftarrow \frac{F_{j-1}}{F_{j-1} \cap F'_1} \longleftarrow \mathbb{H}_{2j-2}(\mathcal{G}_{\mathbb{B}}^2) \xleftarrow{\sigma_{2j}^*} \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1) \xleftarrow{i_{2j}} \frac{\mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1)}{\frac{\text{Ann}_{\mathbb{B}}(L_j \cap L'_1)}{\text{Ann}_{\mathbb{B}}(L'_1)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \longleftarrow 0$$

□

Corollary 5.1. *Let*

$$\left(\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}, \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \right); \left(\begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix}, \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix} \right) \in \langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rangle^{\perp} \subset \mathbb{B}^2 \quad , \quad \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \equiv \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix}, \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \equiv \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix} \in H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$$

be 2 pairs of syzygies which are homologous, then the corresponding small Gobelins constructed with them have equidimensional hyperhomology groups.

Proof. The flags constructed for each pair of syzygies are the same, since they depend only on their homology class in $\mathbb{H}_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$, and hence the hyperhomology groups are equidimensional by Theorem 0.1. □

6 Properties of the flags

Lemma 6.1.

a) *Let $I \subset \mathbb{B}$ be an ideal such that $I = (I \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}$. Then for every $j \geq 0$ we have $L_j \subset I \subset F_j$. F_{∞} is the largest of such class of ideals and L_{∞} is the smallest.*

b) *Let $J \subset \mathbb{B}$ be an ideal such that $J = (J \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}$. Then for every $j \geq 0$ we have $L'_j \subset J \subset F'_j$. F'_{∞} is the largest of such class of ideals and L'_{∞} is the smallest.*

Proof. Let $\phi_1, \phi_2 : \mathbb{B} \rightarrow H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ be the morphisms defined by

$$\phi_1(a) = a \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}, \quad \text{and} \quad \phi_2(a) = a \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}.$$

Note that

$$F_j = \phi_1^{-1} \phi_2(F_{j-1}), \quad L_j = \phi_1^{-1} \phi_2(L_{j-1}), \quad \text{and} \quad 0 = L_0 \subset I = \phi_1^{-1} \phi_2(I) \subset F_0 = \mathbb{B}.$$

We will prove the statement by induction. Clearly, $I \subset \mathbb{B} = F_0$. If $I \subset F_{j-1}$, then $I = \phi_1^{-1} \phi_2(I) \subseteq \phi_1^{-1} \phi_2(F_{j-1}) = F_j$. Hence, $I \subset F_j$ for every $j \geq 0$ and the result follows.

The proof for L_j is analogous because $L_0 = 0 \subset I$. If $L_{j-1} \subset I$, then $L_j = \phi_1^{-1} \phi_2(L_{j-1}) \subset I = \phi_1^{-1} \phi_2(I)$. Hence, $L_j \subset I$ for every $j \geq 0$. \square

Remark 6.1. Lemma 6.1 implies that the chains of ideals stabilize at the first place where there is no proper containment: If $F_j = F_{j+1}$ then $F_\infty = F_j$. Since f_1 and f_2 belong to the ideals F_j, F'_j, L_j and L'_j for $j \geq 1$, we have that the length of all the flags of ideals is smaller than or equal to $\dim_K \mathbb{B}/(f_1, f_2)$.

Lemma 6.2. Let $T_1, T_2 : V \rightarrow W$ be linear transformations of finite dimensional K -vector spaces and $V_2 \subset V_1$ be linear subspaces of V , then

$$\text{codim}(V_2, V_1) \geq \text{codim}(T_1^{-1}T_2(V_2), T_1^{-1}T_2(V_1)).$$

Proof. We have an induced well-defined surjective map $T_2 : \frac{V_1}{V_2} \rightarrow \frac{T_2(V_1)}{T_2(V_2)}$. Then,

$$\dim_K V_1 - \dim_K V_2 \geq \dim_K T_2(V_1) - \dim_K T_2(V_2). \quad (44)$$

We also have an induced well-defined injective map $T_1 : \frac{T_1^{-1}T_2(V_1)}{T_1^{-1}T_2(V_2)} \rightarrow \frac{T_2(V_1)}{T_2(V_2)}$, which gives

$$\dim_K T_2(V_1) - \dim_K T_2(V_2) \geq \dim_K T_1^{-1}T_2(V_1) - \dim_K T_1^{-1}T_2(V_2). \quad (45)$$

The desired inequality follows from (44) and (45). \square

Proposition 6.1. For $j \geq 0$ we have:

$$\begin{aligned} \dim_K \frac{F_j}{F_{j+1}} &\geq \dim_K \frac{F_{j+1}}{F_{j+2}}, & \dim_K \frac{L_{j+2}}{L_{j+1}} &\leq \dim_K \frac{L_{j+1}}{L_j} \\ \dim_K \frac{F'_j}{F'_{j+1}} &\geq \dim_K \frac{F'_{j+1}}{F'_{j+2}}, & \text{and} & \dim_K \frac{L'_{j+2}}{L'_{j+1}} &\leq \dim_K \frac{L'_{j+1}}{L'_j}. \end{aligned}$$

Proof. Let $\phi_1 : B \rightarrow H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ be the morphism defined by $a \rightarrow a \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}$ and let $\phi_2 : B \rightarrow H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ be the morphism defined by $a \rightarrow a \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}$. We notice that $F_j = \phi_1^{-1} \phi_2(F_{j-1})$. Hence,

$$\begin{aligned} \dim_K F_{j+1}/F_j &= \dim_K F_{j+1} - \dim_K F_j = \dim_K \phi_1^{-1} \phi_2(F_j) - \dim_K \phi_1^{-1} \phi_2(F_{j-1}) \\ &\leq \dim_K F_j - \dim_K F_{j-1}. \end{aligned}$$

by Lemma 6.2. Similarly, we obtain the statement about F'_j by interchanging the role of ϕ_1 and ϕ_2 . \square

7 Examples

Recall from Section 3 the notation $\mu := \dim_K \mathbb{B}$ and $\nu := \dim_K \frac{\mathbb{B}}{(f_1, f_2)}$.

7.1 Both syzygies are 0

If $\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in < \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} >^\perp \subset \mathbb{B}^2$, then both flags (8) and (9) are identical, and have the form:

$$L_0 = 0 \subset L_1 = \dots = L_\infty = F_\infty = \dots = F_0 = \mathbb{B}$$

From Proposition 0.1 we obtain that for $j \geq 1$:

$$\mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1) = \frac{\mathbb{B}}{(f_1, f_2)} \quad , \quad \mathbb{H}_{2j-1}(\mathcal{G}_{\mathbb{B}}^1) = H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)) \quad , \quad \mathbb{H}_{2j}(\mathcal{G}_{\mathbb{B}}^1) = \text{Ann}_{\mathbb{B}}(f_1, f_2) \oplus \frac{\mathbb{B}}{(f_1, f_2)}, \quad (46)$$

$\dim \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1) = \nu$ and $\dim \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^1) = 2\nu$ for $j \geq 1$, on using (21). On considering the exact sequences (10) and (11), we see that for $j \geq 1$ the left terms are 0, and the right terms are $\mathbb{H}_*(\mathcal{G}_{\mathbb{B}}^1)$. From this, one obtains readily that $\dim_K \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^2) = (j+1)\nu$.

Hence, the algorithm to construct a $j+2$ cycle from a j cycle in this case, imposes no conditions on the j -cycle, and we have freedom in choosing in a 2ν -dimensional manner. This number bounds the dimension of all hyperhomology groups of small Gobelins with f_1 and f_2 fixed, and with variable pair of syzygies.

If the syzygies satisfy $\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = g_1 \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$, $\begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = g_2 \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$, for $g_j \in \mathbb{B}$, or equivalently their class is 0 in $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$, then by Corolary 5.1 we have the same answer: $\dim_K \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^2) = (j+1)\nu$.

7.2 One syzygy is 0

Proposition 7.1. *Assume that $(c_{11}, c_{12})^t = 0 \in H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ and let τ be the dimension over K of the submodule of $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ generated by $(c_{21}, c_{22})^t$, then the associated flags (8) and (9) are:*

$$0 = L_0 \subset L_1 = \dots = L_\infty = F_\infty = \dots = F_0 = \mathbb{B} \quad (47)$$

$$0 = L'_0 \subset \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \right)_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} = L'_1 = \dots = L'_\infty = F'_\infty = \dots = \dots = F'_1 \subset F'_0 = \mathbb{B}, \quad (48)$$

and for $j \geq 0$ we have $\dim_K \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^2) = \nu + j(\nu - \tau)$.

Proof. (47) and (48) follow from the definitions (8) and (9). In this case, we also have (46) and $\dim \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1) = \nu$ and $\dim \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^1) = 2\nu$ for $j \geq 1$. For $j \geq 1$ we have

$$\frac{F_{j-1}}{F_{j-1} \cap F_1} = \frac{\mathbb{B}}{\left(0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \right)_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}},$$

which has dimension τ . Now $L'_1 = \mathbb{B}$ and for $j \geq 1$ we have

$$L_j = \left(0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \right)_{H^1(\mathcal{K}_{\mathbb{B}}(f_1, f_2)^*)} \subset \mathbb{B},$$

which has dimension $\mu - \tau$. We have

$$\frac{\text{Ann}_{\mathbb{B}}(L_j \cap L'_1)}{\text{Ann}_{\mathbb{B}}(L'_1)} = \text{Ann}_{\mathbb{B}}(L_j)$$

which has dimension τ . Hence in the exact sequences in Theorem 0.1, we see that the proces to construct the $j+2$ hyperhomology group from the j consists in imposing τ restrictions and then we have freedom in choosing a $(2\nu - \tau)$ -dimensional space; so in all the dimension increases by $2(\nu - \tau)$. \square

If we interchange the roles of the 2 syzygies (i.e. interchange the flags) we obtain for this case

$$\dim_K \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^1) = \nu, \quad \dim_K \mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^1) = 2\nu - \tau, \quad \dim_K \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^1) = 2\nu - 2\tau \quad \text{for } j \geq 2, \quad (49)$$

and the invariants

$$\frac{F_{j-1}}{F_{j-1} \cap F'_1} = \frac{\text{Ann}_{\mathbb{B}}(L'_j)}{\text{Ann}_{\mathbb{B}}(L_1)} = 0. \quad (50)$$

Hence, if we construct the $j+2$ hypercohomology classes in this way, there are no restrictions on the j cycle, and we have freedom in choosing of dimension $2(\nu - \tau)$ again.

7.3 One syzygy is in the module generated by the other

If $\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = g \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}$ for some unit $g \in \mathbb{B}$, then the corresponding flags are identical:

$$0 = L_0 \subset L_1 = (0 : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) = L_2 = \dots = L_{\infty} \subset F_{\infty} = \dots = F_0 = \mathbb{B}$$

so that the invariants (50) are also 0, and the computations (49) are also valid in this case, and so we have again have for the same reasons that $\dim_K \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^2) = \nu + j(\nu - \tau)$.

Proposition 7.2. *If $\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = g \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}$ for some non unit $g \in \mathbb{B}$, then the corresponding flags are:*

$$0 = L_0 \subset L_1 = (0 : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) \subset L_2 = (L_1 : g) \subset \dots \subset L_j = (L_1 : g^{j-1}) \subset L_{\infty} = F_{\infty} = \dots = F_0 = \mathbb{B}$$

$$0 = L'_0 \subset L'_1 = (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} = \dots = L'_{\infty} = F'_{\infty} \subset \langle g^j \rangle + L'_1 = F'_j \subset \dots \subset \langle g \rangle + L'_1 = F'_1 \subset F'_0 = \mathbb{B}$$

The flags L_j and F'_j stabilize when $j = \min\{i \in \mathbb{N} : g^i \in L_1\}$ and not before. let τ_2 be the dimension over K of the submodule of $H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))$ generated by $(c_{21}, c_{22})^t$, then for $j \geq 0$ we have $\dim_K \mathbb{H}_j(\mathcal{G}_{\mathbb{B}}^2) = \nu + j(\nu - \tau_2)$.

Proof. For $j \geq 1$ we have $F_j = \mathbb{B}$ directly from the assumption, as well as $L'_j = L'_1$, since $L'_1(\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) = 0$.

We now prove that $F'_j = \langle g^j \rangle + L'_1$ for $j \geq 1$ by induction. We have that

$$a \in F'_1 := (\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} = (g \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}$$

means that there exists $b \in \mathbb{B}$ such that

$$(a - bg) \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = 0 \Leftrightarrow (a - bg) \in (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}) \Leftrightarrow a \in \langle g \rangle + (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}).$$

We assume that our claim is true for j and prove for $j+1$.

$$a \in F'_{j+1} := (F'_j(\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))} = (F'_j g \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})_{H_1(\mathcal{K}_{\mathbb{B}}(f_1, f_2))}$$

means that there exists $b = cg^j + m \in F'_j$, with $c \in \mathbb{B}$, $m \in (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})$. Then,

$$(a - bg) \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = 0 \Rightarrow (a - bg) \in (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}) \Rightarrow (a - cg^{j+1} + mg) \in (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}) \Rightarrow a \in \langle g^{j+1} \rangle + (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}).$$

On the other hand, if $a \in \langle g^{j+1} \rangle + (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})$, there are $c \in \mathbb{B}$, $m \in (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})$ such that $a = cg^{j+1} + m$. Then

$$a \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = cg^{j+1} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} + m \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = cg^j \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \in \mathbb{B} g^j \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} \subset F_j \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}.$$

by induction hypothesis. Therefore, $a \in F_{j+1}$.

Now we prove $L_j = (L_1 : g^{j-1})$ for $j \geq 1$ by induction. We note that it is true for $j = 1$ by the convention $g^0 = 1$. The assumption implies:

$$< \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} > \subset < \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} > \quad , \quad L'_1 := (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}) \subset (0 : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) := L_1 \subset \mathbb{B}. \quad (51)$$

Let $a \in L_{j+1} := (L_j \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) = (L_j \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} : g \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})$, then $(ag - b) \in (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix})$, with $b \in (L_1 : g^{j-1})$. Multiplying by g^{j-1} we obtain

$$ag^j = bg^{j-1} + (0 : \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}) \subset (0 : \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}) \Rightarrow a \in (L_1 : g^j).$$

On the other hand, if $a \in (L_1 : g^j)$ we have that $ag \in (L_1 : g^{j-1}) = L_j$. Then,

$$a \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = ag \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} \in L_j \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}.$$

Therefore $a \in L_{j+1}$.

Directly from Theorem 0.1 we obtain $\dim_K \mathbb{H}_0(\mathcal{G}_{\mathbb{B}}^2) = \nu$ and $\dim_K \mathbb{H}_1(\mathcal{G}_{\mathbb{B}}^2) = 2\nu - \tau_2$.

To prove the rest, we interchange the roles of the 2 syzygies. We have $\frac{\mathbb{B}}{F_1} = 0$, so the flag induced from F'_* is 0. Now $L'_1 \subset L_1 \subset L_j$ from (51), so that the invariants $\frac{L'_1}{L_j \cap L'_1} = 0$. Hence the algorithm to construct the $(j+2)$ -cycles from the j -cycles imposes no conditions and we have freedom of choosing $\mathbb{H}_*(\mathcal{G}_{\mathbb{B}}^1)$, which has dimension $2\nu - 2\tau_2$. \square

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gmont@cimat.mx

Centro de Investigación en Matemáticas, AP 402, Guanajuato, 36000, México

lcn8m@virginia.edu

Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA